Intuitionistic N-Graphs

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Abstract
The geometric system of deduction called N-Graphs was introduced by De Oliveira in 2001. The proofs in this system are represented by means of digraphs and, while its derivations are mostly based on Gentzen’s sequent calculus, the system gets its inspiration from geometrically based systems, such as the Kneales’ tables of development, Statman’s proofs-as-graphs, Buss’ logical flow graphs and Girard’s proof-nets. Given that all these geometric systems appeal to the classical symmetry between premises and conclusions, providing an intuitionistic version of any of these is an interesting exercise in extending the range of applicability of the geometric system in question.

In this paper we produce an intuitionistic version of N-Graphs, based on Maehara’s $LJ'$ system, as described by Takeuti. Recall that $LJ'$ has multiple conclusions in all but the essential intuitionistic rules e.g. implication right and negation right. We show soundness and completeness of our intuitionistic N-Graphs with respect to $LJ'$. We also discuss how we expect to extend this work to a version of N-Graphs corresponding to the intuitionistic logic system $FIL$ (Full Intuitionistic Logic) of de Paiva and Pereira and sketch future developments.

Keywords: Proof theory, proof graphs, N-Graphs, intuitionistic logic, sequent calculus, multiple-conclusion systems.

1 Introduction
The N-Graphs system is a multiple-conclusion proof system for classical propositional logic, developed by de Oliveira (more details can be found in [4], [5]). The N-Graphs system has logical rules corresponding to natural deduction rules and also to some of the structural rules of the sequent calculus.

Proofs in this system are represented by means of digraphs and, while its derivations are mostly based on Gentzen’s sequent calculus, the system gets its inspiration from systems based on graph theory, such as the Kneales’ tables of development [8] [9], Statman’s proofs-as-graphs [13], Buss’ logical flow graphs [3] and Girard’s proof-nets [7].
These graphical frameworks have in common the idea of a geometric perspective, where proofs are studied in terms of their structural properties.

The study of structural properties of proofs involves questions like how long is a proof of a given formula, how to identify two proofs of the same theorem, what happens if we invert a given derivation, what is the meaning of this operation, etc. A famous general result about the structure of proofs is the cut-elimination theorem (the *Hauptsatz*) for sequent calculus given by Gentzen in [6] as well as its corresponding normalization procedure for Natural Deduction (ND) systems given by Prawitz in [11].

Given that all these geometric systems appeal to the classical symmetry between premises and conclusions, providing an *intuitionistic* version of any of these is a hard task. Putting it more constructively it is an interesting exercise in extending the range of applicability of the geometric system in question.

The multiple-conclusion system N-Graphs is essentially classical, that is, it did not have a version for intuitionistic propositional logic, as originally conceived. (There are also no intuitionistic versions of tables of development, proofs-as-graphs or logical flow graphs, as far as we know.) We propose a way of constructing a proof system akin to N-Graphs, but tailored to intuitionistic logic.

In Gentzen’s original work [6] the difference between the intuitionistic system, the calculus $LJ$ and the classical system, the calculus $LK$, is the cardinality restriction of the succedent of the sequents. The formulation of the system $LK$ uses sequent expressions of the form $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are sequences of formulas. One such sequent is intuitively read as $A_1 \land \ldots \land A_n \rightarrow B_1 \lor \ldots \lor B_m$, the conjunction of the premisses entails the disjunction of the conclusions. In the calculus for intuitionistic logic, $LJ$, sequents are restricted to succedents with at most one formula occurrence, usually represented by $\Gamma \vdash B$. However, there are several well-known multiple-conclusion systems for intuitionistic logic.

First we recall the system $LJ'$ proposed by Maehara, which is described in Takeuti’s book [14]. This system is obtained by setting restrictions on the calculus $LK$ (instead of $LJ$) in the two rules of inference *negation on the right* ($R\neg$) and *implication on the right* ($R\rightarrow$), as presented below. All other rules of inference are the same as the rules of the $LK$ system. (Note that since negation can be defined in terms of falsehood as $\neg A = A \rightarrow \bot$, only the modification on the implication rule is essential.)

\[
\frac{D, \Gamma \vdash \neg D}{\Gamma \vdash \neg D} \quad R\neg
\]

\[
\frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \quad R\rightarrow
\]

Another multiple conclusion proof system for propositional intuitionistic logic was developed by de Paiva and Pereira, the system $FIL$ [10]. This system uses a indexing device in the sequent, which allows tracking of the dependency relations between formulas in the antecedent and in the succedent of the sequent. A condition in the *right implication* rule ($R\rightarrow$) ensures that only valid constructive formulas are derived. The system $FIL$ has the advantage of being closer to true multiple-conclusions as used in classical logic, but still retains some form of constructive constraint, otherwise it would be able to derive all theorems of classical logic.

In this work we choose the system $LJ'$ to base our intuitionistic N-Graphs on. We first present a description of N-Graphs in section 2. In section 3, we show a description of N-Graphs for intuitionistic logic based on $LJ'$ and how to prove the soundness and completeness for these intuitionistic N-Graphs. In section 4 we conclude with
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2 N-Graphs

In this section we introduce N-Graphs, the formal system developed by de Oliveira [4]. We begin with basic definitions of N-Graphs then we see how to build an N-Graph derivation and we describe the soundness criterion. Finally, we present two examples of derivations in the proof system.

2.1 Definitions

N-Graphs are directed graphs that represent proofs where each vertex is an occurrence of a formula and each edge represents an atomic step in a derivation. Proofs are represented by graphs which are constructed from a set of basic links, as illustrated in Fig. 1. Basic links represent schematically the rules of the calculus.

The propositional language uses logical constants: \( \land \) (conjunction), \( \lor \) (disjunction), \( \rightarrow \) (implication), \( \neg \) (negation), the constant \( \bot \) for absurdity (or false) and the constant \( \top \) for truth. We use the letters \( A, B, C, D, E, \ldots \) for arbitrary formulas (or formula-occurrences) in the language.

![Fig. 1. N-Graphs links](image-url)
The essence of the definition of N-Graphs is the way we deal with inference rules in the graphical formalism. This is described in Fig. 1. But we also need some auxiliary notions, as follows:

**Definition 2.1 (Focussing/Defocussing branch point)**

A *focussing branch point* is a vertex in a digraph with two edges oriented towards it (See Fig. 2). A *defocussing branch point* is a vertex in a digraph with two edges oriented away from it.

**Definition 2.2 (Focussing/Defocussing/Simple links)**

A *focussing link* is a set \( \{(u_1, v), (u_2, v)\} \) in a digraph in which \( v \) is a focussing branch point as illustrated in Fig. 2. The vertices \( u_1 \) and \( u_2 \) are called premisses of the link, while the vertex \( v \) is the conclusion. A *defocussing link* is a set \( \{(u, v_1), (u, v_2)\} \) in a digraph in which \( u \) is a defocussing branch point as illustrated in Fig. 2. The vertices \( v_1 \) and \( v_2 \) are called conclusions of the link, while \( u \) is the premise. A *simple link* is an edge \((u, v)\) in a digraph which neither belongs to a focussing nor to a defocussing link. The vertex \( u \) is called the premise of the link and \( v \) its conclusion.

![Fig. 2. Kinds of links](image)

N-Graphs are somewhat similar to proof-nets. The same way we first define proof structures and then provide a criterion to determine which proof structures are logically sound, hence proof-nets, we first define proof-graphs, define a criterion to determine which proof-graphs are logically sound and then we call these the N-graphs.

**Definition 2.3 (Proof-graph)**

A *proof-graph* is a connected oriented graph defined as follow:

1. Each vertex is labelled with a formula-occurrence.
2. The edges are of two kinds (“meta” and “solid”), the meta-edges are labelled by a letter “m” \( (u, v)^m \), all the other edges are called solid edges (See example Fig. 3). The meta-edges are used to indicate the cancellation of the hypothesis.
3. There are three kinds of links: simple, focussing and defocussing, divided into logical and structural ones.
4. Every vertex in a proof-graph is labelled with a conclusion of a unique link and is the premise of at most one link.

The set of edges in a proof-graph may be empty. In this case the proof-graph represents an axiom.

In Fig. 1, we present the logical and structural links of N-Graphs. And bellow we present some definitions concerning N-Graphs links.

**Definition 2.4 (Conjunctive link)**

The links \( \land I, \neg E, \rightarrow E, \top -focussing-weakening \) and *expansion* link are called conjunctive.
Definition 2.5 (Disjunctive link)
The links \( \lor -E \), \( \neg -I \), \( \rightarrow -I \), \( \bot -\text{defocussing-weakening} \) and \( \text{contraction} \) link are called disjunctive.

Note that all disjunctive links with the exception of the contraction link are defocussing ones as well as all conjunctive links are focussing with the exception of the expansion link. We will also need the following definition:

Definition 2.6 (Solid indegree/outdegree of v in a proof-graph)
The solid \( \text{indegree} \) of a vertex \( v \) in a proof-graph is the number of solid edges oriented towards it. The solid \( \text{outdegree} \) of a vertex \( v \) in a proof-graph is the number of solid edges oriented away from it.

The expansion and contraction links (under Focussing/Defocussing structural links in Fig. 1) are also called switching links. And their edges are respectively called switching edges. The set of vertices with outdegree equal to zero is the set of conclusions of a derivation represented in a proof-graph \( G \) and is written as \( \text{CONC}(G) \). And the set of vertices with indegree equal to zero is the set of premisses of \( G \), and it is written as \( \text{PREM}(G) \). The set of hypotheses of the graph \( \text{HYPO}(G) \) is the set of vertices in \( G \) with solid indegree equal to zero, but meta indegree equal to 1, denoting the set of cancelled hypothesis.

2.2 Soundness criterion

In order to determine whether an N-Graph corresponds to a logically correct proof-graph, one must remove an edge from every link of expansion and contraction (see Fig. 1) in this graph. Thus, new proof-graphs are generated, which must be acyclic and connected to be a derivation logically valid. Below we recall the geometric criteria formally established by Oliveira [4] to test whether a proof-graph is logically correct:

Definition 2.7 (Switching graph)
Given a proof-graph \( G \), a switching graph \( S \) associated with \( G \) is a spanning subgraph\(^1\) of \( G \) in which the following edges are removed:

- one of the two edges of every expansion and contraction link;
- all meta-edges.

Definition 2.8 (Switching expansion graph)
Given a proof-graph \( G \), a switching expansion graph \( S' \) associated with \( G \) is a spanning subgraph of \( G \) in which all meta-edges are removed as well as one of the two edges of every expansion link is removed.

Definition 2.9 (Meta-condition)
Given a proof-graph \( G \), we say that the meta-condition holds for it iff for every meta-edge \((u, v)^m\) of a defocussing link \( \{(u, v)^m, (u, w)\} \) in \( G \), there is a path\(^2\) or a semipath\(^3\) from \( v \) to \( u \) which belongs to every switching expansion graph \( S' \) associated with \( G \).

---

1 A spanning subgraph is a subgraph \( G_1 \) of \( G \) containing all the vertices of \( G \).
2 A path \( v_1, v_2, \ldots, v_n \) in a graph \( G \) is a sequence of distinct vertices \( v_1, v_2, \ldots, v_n \) with \( n > 1 \), and \( (v_i, v_{i+1}) \) an edge for \( 1 \leq i < n \).
3 A semipath in a digrah is an alternating sequence of distinct vertices and edges \( v_0, x_1, v_1, \ldots, x_n, v_n \) in which each edge \( x_i \) may be either \((v_{i-1}, v_i)\) or \((v_i, v_{i-1})\).
and that path cannot pass through \((u, w)\). Moreover, the solid indegree of \(v\) in \(G\) is equal to zero.

**Fig. 3:** An unsound proof-graph with a meta-edge  
**Fig. 4:** A sound proof-graph with a meta-edge

**Definition 2.10 (N-Graphs derivation)**  
A proof-graph \(G\) is an N-Graph derivation, or N-Graph for short, iff the meta-condition holds for \(G\) and every switching graph associated with \(G\) is acyclic and connected.

Looking at the examples shown in Fig. 3 and 4, we realize that in the graph of the Fig. 3 we have the formula \(A \lor B\) labelling the vertex \(u\), \(A\) labelling the vertex \(v\) and \(A \rightarrow A \lor B\) the vertex \(w\). Thus, applying the meta-condition, we notice that \(v\) has solid indegree equal to zero. However, there is a switching graph expansion associated with this graph where there is no path or semipath from \(v\) to \(u\) not passing via \((u, w)\). This happens because the only potential path is formed by an expansion link, as we know we shall remove one of the two edges of every expansion link. As a result, this graph does not satisfy the meta-condition. Now inspecting the graph of the Fig. 4, we check that in the above meta-edge we have the formula \((A \lor B) \land (A \lor C)\) as \(u\), \(A \lor (B \land C)\) as \(v\) and \(A \lor (B \land C) \rightarrow ((A \lor B) \land (A \lor C))\) as \(w\). Applying the meta-condition, we determine that \(v\) has solid indegree equal to zero and obtain a straight path from \(v\) to \(u\). As a consequence, this graph does satisfy the meta-condition and it is indeed an N-Graph derivation.

### 3 Intuitionistic N-Graphs

We now propose a version of N-Graphs for intuitionistic logic based on the sequent calculus \(LJ'\) of Maehara and prove soundness and completeness for this system.

We use the same propositional language of N-Graphs, that is, the logical connectives \(\land\) (conjunction), \(\lor\) (disjunction), \(\rightarrow\) (implication), \(\neg\) (negation), and the constants \(\bot\) for *absurdity* (or *falsehood*) and \(\top\) for *truth*. 
We also use the set of links of the N-Graphs (see Fig. 1) that represent the atomic steps in a derivation, but clearly we need to modify them somewhat to ensure compliance with the intuitionistic features of the system.

Intuitionistic logic is a sublogic of classical logic, as in classical logic we can draw inferences that we cannot in intuitionistic logic. Specifically, intuitionistic logic rejects the principle of excluded middle ($A \lor \neg A$). This principle is clearly represented in N-Graphs by the logical $\top$-link (see Fig. 1), so we remove it and introduce a new link $\neg I$. More importantly, we establish a restriction in the links $\neg I$ and $\rightarrow I$ described below.

First we recall that just like in N-Graphs, the construction of an intuitionistic N-Graphs derivation follows the definition of a proof-graph (see Definition 2.3). It uses the set of logical and structural links illustrated in Fig. 6. Note the restriction on the links $\neg I$ and $\rightarrow I$. When applying those links, $\bot$ or $B$ must be the only conclusion at the time of its addition to the intuitionistic N-Graph $G$ under construction. This restriction is similar to that of the calculus $LJ'$, which restricts the rules $R\neg$ and $R\rightarrow$.

Consider the attempt to build an intuitionistic N-Graph for the law of excluded middle in Fig. 5, the graph clearly does not satisfy the restriction imposed on the link $\neg I$. Thus, at the time that you apply the link $\neg I$, we see that $\bot$ is not the only conclusion, then the link $\neg I$ should not be applied.

![Fig. 5. An unsound proof-graph](image)

Proof-graphs, like proof structures in the case of proof-nets, can be unsound. To determine the logically correct proof-graphs, the true intuitionistic N-Graphs, we need a global soundness criterion (see [4] and [1]). These problems seem to come from the multiple-conclusion structure of the graphs, and may lead to derivations with cycles that will turn out to be logically unsound (see [2]). Thus we need a global soundness criterion to determine whether the proof-graph is, in fact, an intuitionistic N-Graph derivation or not.

3.1 Soundness criterion

The global soundness criterion for intuitionistic N-Graphs is obtained by modifying appropriately the definition of the soundness criterion for classical N-Graphs, that is Definitions 2.9 and 2.10 in Section 2.2. The other definitions will remain the same.

**Definition 3.1** (Intuitionistic meta-condition)

Given a proof-graph $G$, we say that the intuitionistic meta-condition holds for it iff for every meta-edge $(u,v)^m$ of a defocussing link $\{(u,v)^m,(u,w)\}$ in $G$:

i) there is a path or a semipath from $v$ to $u$ which belongs to every switching expansion graph $S'$ associated with $G$, and that path cannot pass through $(u,w)$. Moreover, the solid indegree of $v$ in $G$ is equal to zero;
ii) in the resulting graph $S$ of the removal of all meta-edges, every path that starts from any vertex in $\text{PREM}(S)$ that goes to any vertex in $\text{CONC}(S)$ must pass by $(u, v)$.

**Definition 3.2 (Intuitionistic N-Graph derivation)**

A proof-graph $G$ is an intuitionistic N-Graph derivation iff the intuitionistic meta-condition holds for $G$ and every switching graph associated with $G$ is acyclic and connected.

Consider the two examples of an intuitionistic N-Graph derivation. The first one side-by-side with a corresponding sequent derivation is shown in Table 1 and the second example is shown in the Fig. 7. In the graph of the Table 1, we have formulas labelling the vertices $s, t, u, v, w, x, y$ and $z$. Thus, applying the intuitionistic meta-condition, we notice that $s, t$ and $w$ has solid indegree equal to zero. There is no switching graph expansion associated with this graph. In the resulting graph the removal of all meta-edges, every path that starts from the vertex $s$ that goes to the vertex $z$ passes by $(s, x)$, every path that starts from the vertex $t$ that goes to the vertex $z$ passes by $(t, u)$ and every path that starts from the vertex $w$ that goes to the vertex $z$ passes by $(w, z)$. As a result, this graph satisfies the intuitionistic meta-
condition and it is an intuitionistic N-Graph derivation. The second example has no meta-edges and it is an intuitionistic N-Graph derivation.

<table>
<thead>
<tr>
<th>Table 1. Intuitionistic N-Graph of ( \neg \neg \neg A \rightarrow \neg A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivation in ( LJ' )</td>
</tr>
</tbody>
</table>
| \[
| \begin{array}{c}
| A \vdash A \quad L\neg \\
| A, \neg A \vdash \neg A \quad R\neg \\
| A, \neg \neg A \vdash \neg A \quad L\neg \\
| \neg \neg \neg A \vdash \neg A \quad R \rightarrow \\
| \vdash \neg \neg \neg A \rightarrow \neg A \\
| \end{array}
| \]

Fig. 7. Intuitionistic N-Graphs of \( A \lor B, (A \rightarrow Z) \lor B \vdash Z, B \land (A \rightarrow Z), B \)

3.2 Soundness and completeness for Intuitionistic N-Graphs

In order to ensure that every intuitionistic N-Graph derivation represents a logically correct proof we need to prove soundness and completeness of the system. Soundness is proved in Theorem 3.4, while completeness is established in Theorem 3.3.

**Theorem 3.3** (Map to intuitionistic N-Graphs)

Given a derivation \( \pi \) of \( A_1, ..., A_n \vdash B_1, ..., B_m \) in the \( LJ' \) calculus, it is possible to build a corresponding intuitionistic N-Graph \( NG(\pi) \) whose elements of \( PREM(NG(\pi)) \) and \( CONC(NG(\pi)) \) are in one-to-one correspondence with the occurrences of formulae \( A_1, ..., A_n \) and \( B_1, ..., B_m \) respectively.

The analogous result for classical N-Graphs was proved by De Oliveira in [4]. Here we adapt her proof to the intuitionistic case.

**Proof.** We prove completeness by induction on the structure of the derivation \( \pi \) in the calculus \( LJ' \), whose equivalence with \( LJ \) was proved by Maehara. We follow the
procedure given by Oliveira in [4], but we present only the cases where the systems differ for this theorem. When the antecedent (succedent) of the derivation sign ⊢ is empty we use the constant T (∲).

• **Case 3:** If π is obtained from π₁ by R¬:

\[
\begin{align*}
\pi_1 & \\
A_2, \ldots, A_n, A & \vdash \quad R \quad \rightarrow
\end{align*}
\]

NG(π) looks like the graph in Fig. 8.

![NG(π) case 3](image)

By induction hypothesis we have obtained NG(π₁). NG(π) is obtained by adding the vertices labeled ∲ and ¬A as well as the meta-edge (∲, A) and the edge (∲, ¬A) to NG(π₁).

Since NG(π) is an iN-Graph, the intuitionistic meta-condition should hold. It is obvious that the solid indegree of A is equal to zero and ∲ is the only conclusion in NG(π₁), however the other conditions need to be analyzed carefully:

1. A and ∲ are linked in NG(π₁) (through a path or semi-path). Otherwise, NG(π₁) would be disconnected;
2. If A is a branch point of an expansion link, then both conclusions of this link are also connected to ∲ (through a path or semi-path). Otherwise, a switching graph associated with NG(π₁) would be disconnected;
3. Then we conclude that in every switching expansion graph associated with NG(π) there is a path or semi-path from A to ∲.

• **Case 8:** If π is as follows:

\[
\begin{align*}
\pi_1 & \\
A_1, \ldots, A_n, A & \vdash B \\
A_1, \ldots, A_n & \vdash A & \rightarrow & B & R & \rightarrow
\end{align*}
\]

NG(π) looks like graph in Fig. 9.

For NG(π) be an iN-Graph, the intuitionistic meta-condition should hold. It is obvious that the solid indegree of A is equal to zero and that B is the only conclusion in NG(π₁), however the other conditions need to be considered carefully:

1. A and B are linked in NG(π₁) (through a path or semi-path). Otherwise, NG(π₁) would be disconnected;
2. if $A$ is a branch point of one expansion link, then both conclusions of this link are also linked to $B$ (through a path or semi-path). Otherwise, a switching expansion graph with $\text{NG}(\pi_1)$ would be disconnected.

3. Then we conclude that in every switching expansion graph associated with $\text{NG}(\pi)$ there is a path or semi-path from $A$ to $B$.

**Theorem 3.4 (Sequentialization)**

Given an intuitionistic N-Graph derivation $G$, there is a derivation in the intuitionistic sequent calculus $LJ' \text{SC}(G)$ of $A_1,\ldots,A_n \vdash B_1,\ldots,B_m$, whose occurrences of formulae $A_1,\ldots,A_n$ and $B_1,\ldots,B_m$ are in one-to-one correspondence with the elements of sets $\text{PREM}(G)$ and $\text{CONC}(G)$ respectively.

**Proof.** Following the procedure given by Oliveira in [4] due to the similarity of the proof, we present only the different case for this theorem. We need the following definition.

**Definition 3.5 (Final $\rightarrow$I ($\neg$-I))**

A link $\rightarrow$I ($\neg$-I) $\{(u,v_1)^m,(u,v_2)\}$ in a given iN-Graph $G$ is final if $v_2 \in \text{CONC}(G)$.

The proof of the Theorem 3.4 proceeds by induction on the number of vertices of a given iN-Graph $G$. We provide an algorithm to transform an iN-Graph $G$ in a derivation $\pi$ of the corresponding calculus $LJ'$.

The proof of the case is as follows:

- **Case 3:** If the link $\rightarrow$I or $\neg$-I $\{(u,v_1)^m,(u,v_2)\}$ is final. Then, removing it from $G$, i.e. the edges $(u,v_1)^m$ and $(u,v_2)$ as well as the vertex $v_2$, two subgraphs are obtained, otherwise there would not be a path or semi-path from $v_1$ to $u$, they are:
  i) the vertex $v_2$;
  ii) the iN-Graph $G_1$ with $v_1 \in \text{PREM}(G_1)$ and $u \in \text{CONC}(G_1)$. $G_1$ is clearly an iN-Graph. It is illustrated in the dotted box of Fig. 9 for the case $\rightarrow$I and in the Fig. 8 for the case $\neg$-I.

The induction hypothesis, in the case of $\rightarrow$I, has built $\text{CS}(G_1)$ as follows:

$$\pi_1$$


\[ A_1,\ldots,A_n, A \vdash B \]
Therefore, $CS(G)$ is deduced from $CS(G_1)$ by $R \rightarrow$:

$$
\pi_1
\frac{A_1, \ldots, A_n, A \vdash B}{A_1, \ldots, A_n \vdash A \rightarrow B} R \rightarrow
$$

The other case is as follows:

$$
\pi_1
\frac{A_1, \ldots, A_n, A \vdash}{A_1, \ldots, A_n \vdash \neg A} R \neg
$$

Therefore, $CS(G)$ is deducted from $CS(G_1)$ by $R\neg$:

$$
\pi_1
\frac{A_1, \ldots, A_n, A \vdash}{A_1, \ldots, A_n \vdash \neg A} R \neg
$$

4 Conclusions and future work

We extended N-Graphs that were proposed for classical propositional logic, to a system that caters for intuitionistic propositional logic. N-Graphs are a multiple-conclusion system, so we rely on a basic intuitionistic sequent calculus with multiple-conclusions. In this paper we used Maehara’s system $LJ'$ as our basic multiple conclusion intuitionistic sequent calculus, as it seemed the closest to the original N-Graphs.

We have proved soundness and completeness for this system, which we call $iN$-Graphs, for intuitionistic N-Graphs, by a modification of the original proof for classical N-Graphs. In future work we intend to describe and prove sound and complete intuitionistic N-Graphs based on the system $FIL$ of Full Intuitionistic Logic, introduced by de Paiva and Pereira, [10] mentioned in the introduction. While the system $LJ'$ is a sound basis for our constructive geometric explorations, its ability to deal with cut-free proofs is somewhat limited, as its process of cut-elimination throws derivations out of the system([12]). We hope to obtain better behavior of the process of cut-elimination (and of its geometric features) using the system $FIL$. We have a draft of an intuitionistic system for N-Graphs with a indexing device that allows us to record the premisses on which each formula in the graph depends on, and thus to correctly apply the links $\neg I$ and $\rightarrow I$ and determine the soundness of the intuitionistic N-Graphs. We construct the proof of completeness of the system by mapping the intuitionistic N-Graphs to $FIL$, and we hope to be able to prove soundness of these new intuitionistic n-graphs by sequentializing proofs in $FIL$. But this remains future work. Also future work is to show how one can normalize $iN$-Graphs and how this normalization relates to cut-elimination in the sequent calculus.

References

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